On the Long Run Volatility of Stocks [1]

Carlos M. Carvalho
McCombs School of Business
The University of Texas

[1] with Hedibert Lopes and Rob McCulloch.

Questions

- What is the long-run variance of stocks returns?

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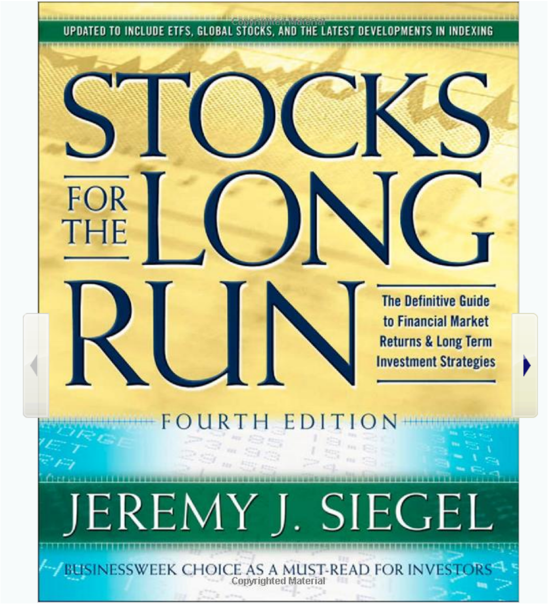
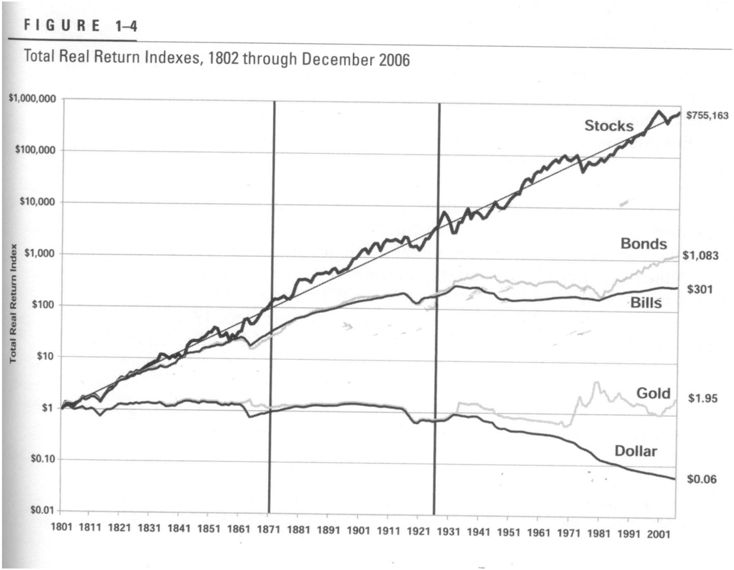
- Should the long-horizon investor allocate his wealth differently from a short-horizon investor? Why?

Stocks for the Long Run

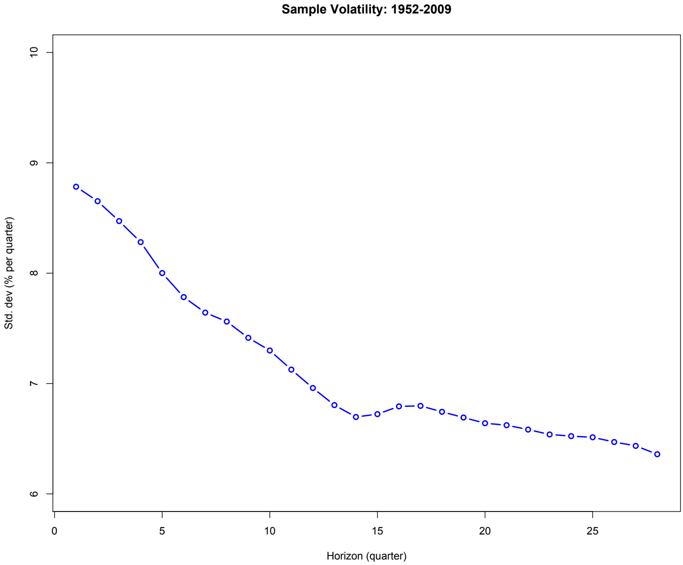


Stocks for the Long Run: Conventional Wisdom



Stocks for the Long Run: Conventional Wisdom

- x-axis: Horizon y-axis: Volatility per quarter



Summary

Taking a conditional approach from the investor's perspective:

- A simple view of the world tells us that "stocks are good" for long horizons (Barberis, 2000)...
- ...while a very complex view of the world states that "stocks are bad" for long horizons (Pastor and Stambaugh, 2011)

- Working with alternative, more parsimonious models that preserve the economic motivation of PS2011 we find that stocks "are indeed good" for the long-run.

Background

- Returns k periods in the future:

$$r_{1,k} = r_1 + r_2 + r_3 + \cdots + r_k$$

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$$r_{1,k} = r_1 + r_2 + r_3 + \cdots + r_k$$

- If returns are i.i.d. $r_i \sim N(\mu, \sigma^2)$, i.e., *random walk* on prices:

$$Var(r_{1,k}) = k\sigma^2$$

so that the variance per period is constant for any k investment horizon.

Background

- However, investor face parameter uncertainty...
- If $r_t \sim N(\mu, \sigma^2)$ and μ is unknown then,

$$
\begin{aligned}
Var(r_{t,t+k}|D_t) &= \mathrm{E}\left\{Var(r_{t,t+k}|\mu, D_T)\right\} + Var\left\{\mathrm{E}(r_{t,t+k}|\mu, D_T)\right\} \\
&= k\sigma^2 + k^2 Var(\mu|D_t)
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- Long run volatility grows linearly with the horizon
- Is conventional wisdom wrong? No necessarily!

Background

- If μ is mean reverting and

$$\begin{aligned} r_{t+1} &= \mu_t + u_{t+1} \\ \mu_{t+1} &= \alpha + \beta\mu_t + w_{t+1} \end{aligned}$$

where $Corr(u_{t+1}, w_{t+1}) < 0$,

$$Var(r_{t,t+k}) = k\sigma^2 \left[1 + 2A\rho_{uw} + B\right]$$

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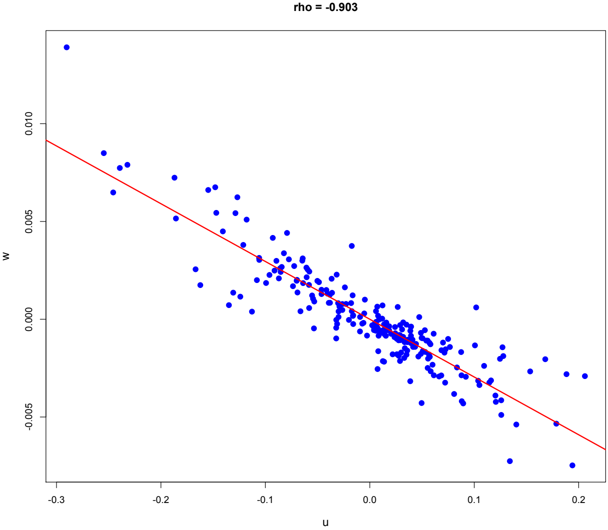
$$Var(r_{t,t+k}) = k\sigma^2 \left[1 + 2A\rho_{uw} + B\right]$$

- The effect of ρ_{uw} effect can dominate and imply a decreasing long run risk as both $A > 0$ and $B > 0$.

Background

- For stocks... using dividend yield as a proxy for expected returns:



Today

1. Alternative models specifications... priors, time variation...

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2. Some preliminary results

The Starting Point

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r_{t+1} &= \mu_t + u_{t+1} \\
\mu_{t+1} &= \alpha + \beta \mu_t + w_{t+1}
\end{aligned}
$$

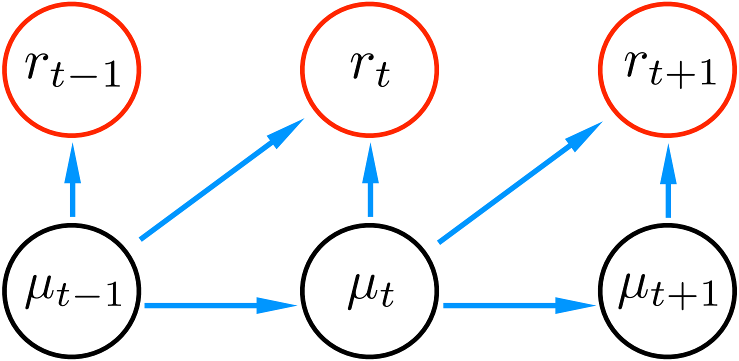
where

$$
\begin{pmatrix} u_{t+1} \\ w_{t+1} \end{pmatrix} \sim N(0, \Sigma)
$$

and

$$
\mathrm{E}(r_{t+1} | \mathcal{D}_t) = \mu_t
$$

The Starting Point



Predictive Regressions (Barberis, 2000)

$$
\begin{aligned}
r_{t+1} &= a + b\, x_t + u_{t+1} \\
x_{t+1} &= \alpha + \beta x_t + w_{t+1}
\end{aligned}
$$

where

$$
\begin{pmatrix} u_{t+1} \\ w_{t+1} \end{pmatrix} \sim N(0, \Sigma)
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- "Perfect predictors" imply decreasing variance for stocks in the long run even in the presence of parameter uncertainty.

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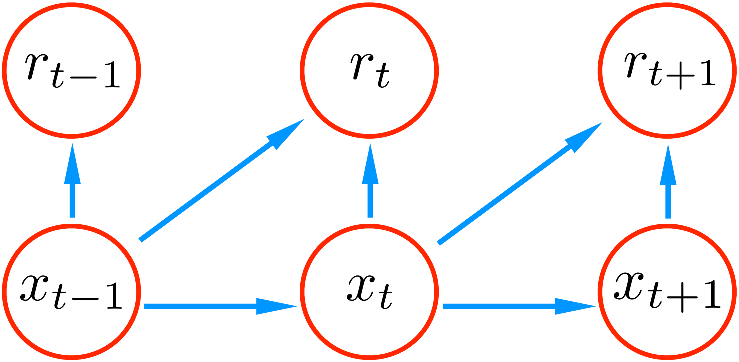


Predictive Systems (Pastor and Stambaugh, 2011)

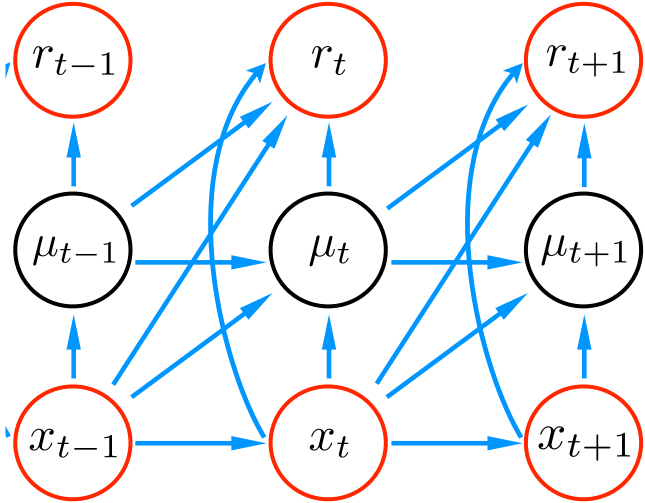
$$\begin{aligned} r_{t+1} &= \mu_t + u_{t+1} \\ \mu_{t+1} &= \alpha + \beta\mu_t + w_{t+1} \\ x_{t+1} &= A + Bx_t + v_{t+1} \end{aligned}$$

where

$$\begin{pmatrix} u_{t+1} \\ w_{t+1} \\ v_{t+1} \end{pmatrix} \sim N(0, \Sigma)$$

- "Imperfect Predictors:" once all the uncertainty is accounted for, stocks are riskier in long investment horizons.

Predictive Systems (Pastor and Stambaugh, 2011)



Our Goals

- Maintain the economic motivation while work with more parsimonious specifications
  1. uncertainty about predictors
  2. negative correlation between expected and realized returns

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- Develop a better understanding of the impact of priors in this very low signal environment.

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- Maintain the economic motivation while work with more parsimonious specifications
  1. uncertainty about predictors
  2. negative correlation between expected and realized returns
- Add complexity and allow for time varying relationships and volatilities
- Develop a better understanding of the impact of priors in this very low signal environment.
- In summary, we need a “simple” and “encompassing” modeling framework that is time varying, embeds our prior knowledge and is computable.

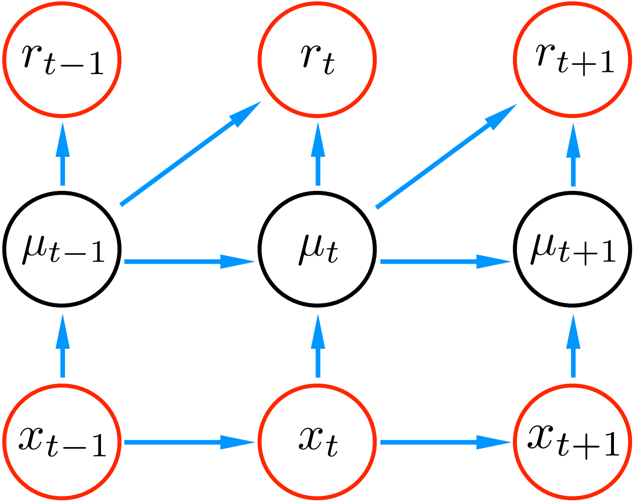
Proposal

$$
\begin{aligned}
r_{t+1} &= \mu_t + u_{t+1} \\
\mu_{t+1} &= \alpha + \beta\mu_t + x'_{t+1}\gamma + w_{t+1} \\
x_{t+1} &= A + Bx_t + v_{t+1}
\end{aligned}
$$

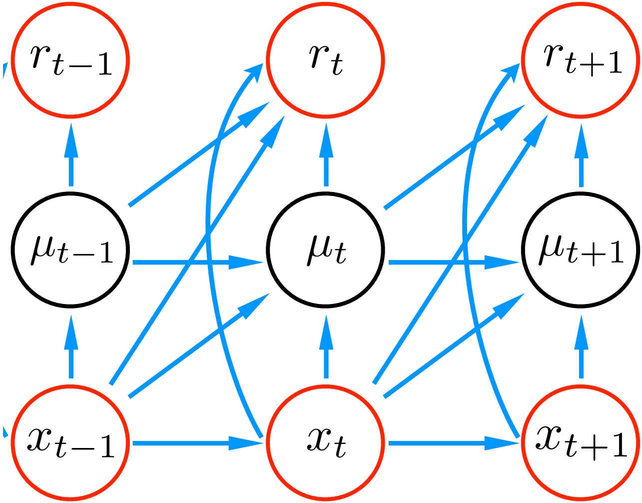
where

$$
\begin{pmatrix} u_{t+1} \\ w_{t+1} \\ v_{t+1} \end{pmatrix} \sim N\left(0, \begin{bmatrix} \Sigma_{t+1} & 0 \\ 0 & \Sigma^X_{t+1} \end{bmatrix}\right)
$$

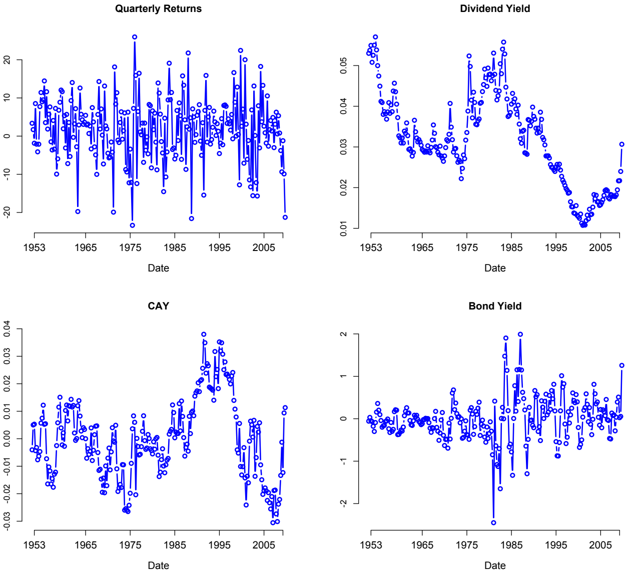
Proposal



Predictive Systems

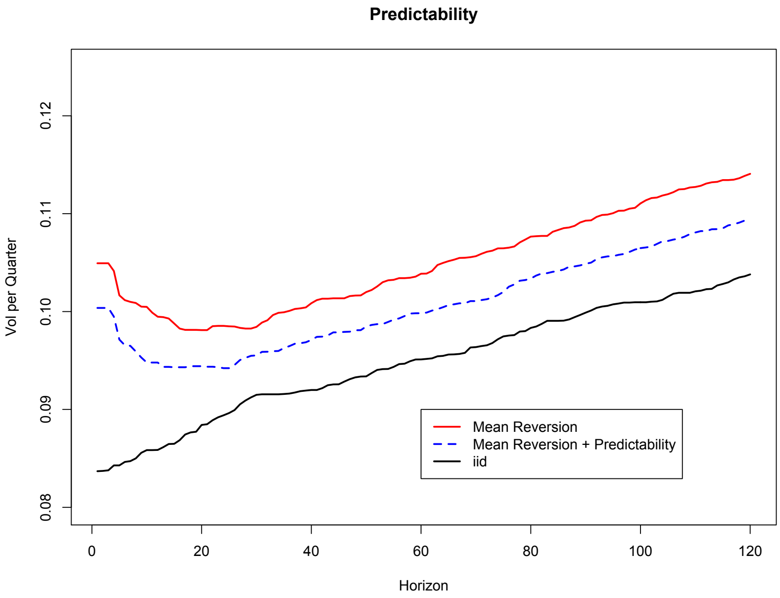


Data

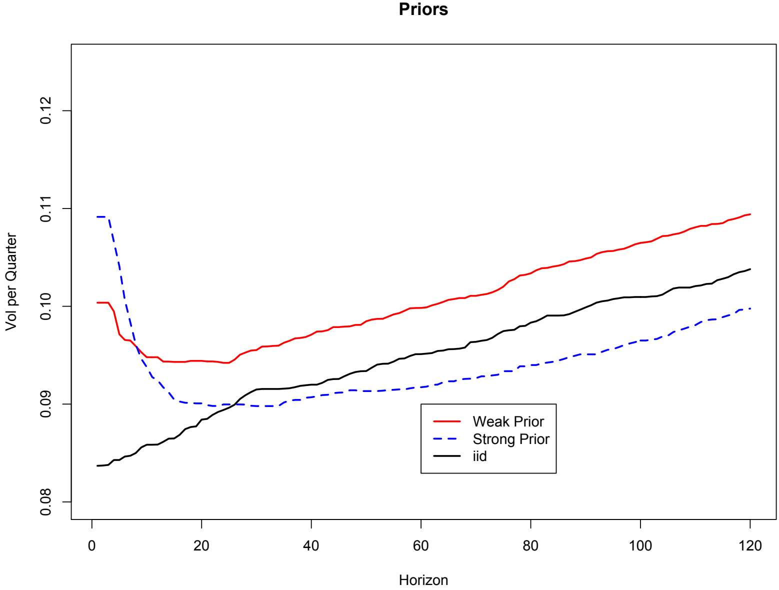


Quarterly Returns
20
10
0
-10
-20
1953
1965
1975
1985
1995
2005
Date
Dividend Yield
0.05
0.04
0.03
0.02
0.01
1953
1965
1975
1985
1995
2005
Date
CAY
0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
-0.03
1953
1965
1975
1985
1995
2005
Date
Bond Yield
2
1
0
-1
-2
1953
1965
1975
1985
1995
2005
Date

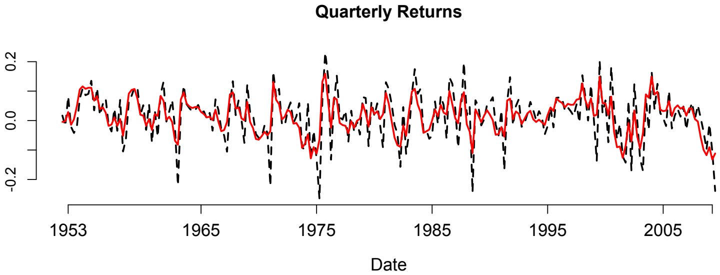
Preliminary Results – Predictability

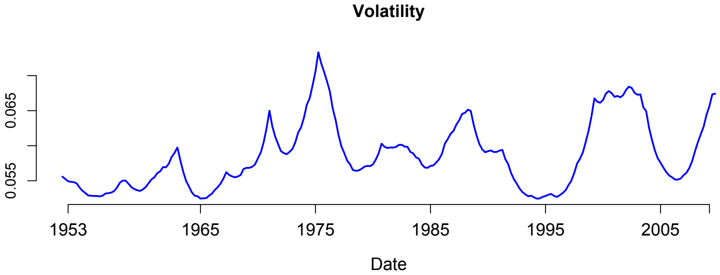


Preliminary Results – Priors



Preliminary Results - Time Varying Covariance Matrix





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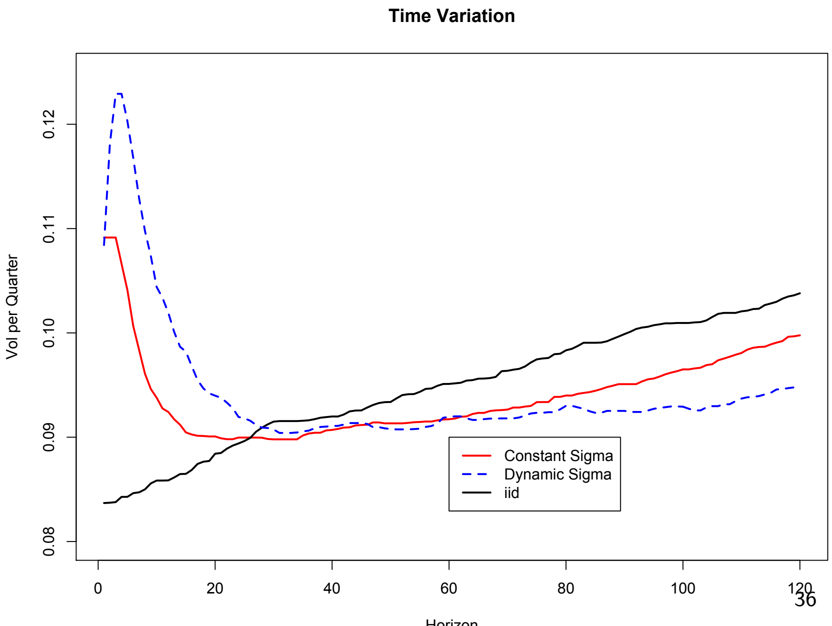

Early Conclusions...

- With reasonable priors about the stationary distribution on the equity premium, the long run volatility of stocks is smaller than what would be implied by "efficient markets".

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- The implication: unless you have some crazy beliefs...

Early Conclusions...

- With reasonable priors about the stationary distribution on the equity premium, the long run volatility of stocks is smaller than what would be implied by "efficient markets".
- In fact, we reasonably informative priors, the iid model is an upper bound for long run variance.
- The implication: unless you have some crazy beliefs...
  BUY MORE STOCKS!!

Time Variation

Instead of just Σ, we want Σ_t, *and* we want to easily incorporate the prior belief that

$$\rho_t = corr(u_t, w_t) < 0, \text{for all } t$$

and possibly other prior beliefs as well.

Multivariate Stochastic Volatility

We start with the Choleski Stochastic Volatility approach of Lopes, McCulloch, and Tsay.

With one x we have:

$$
\begin{array}{rcll}
w_t & = & \exp(\theta_{t1}/2)\, Z_{t1} & p(w_t) \\
u_t & = & \theta_{t3}\, w_t + \exp(\theta_{t2}/2)\, Z_{t2} & p(u_t \mid w_t) \\
v_t & = & \phi_{t2}\, w_t + \phi_{t3}\, u_t + \exp(\phi_{t1}/2)\, Z_{t3} & p(v_t \mid w_t, u_t)
\end{array}
$$

At each t, the three θ's and three ϕ's are one to one with Σ_t.

Let's just focus on the θ's because they determine ρ_t.

Multivariate Stochastic Volatility

We have,

$$
\begin{aligned}
w_t &= \exp(\theta_{t1}/2)\, Z_{t1} \\
u_t &= \theta_{t3}\, w_t + \exp(\theta_{t2}/2)\, Z_{t2}
\end{aligned}
$$

$$
\rho_t = \rho(\theta_{t1}, \theta_{t2}, \theta_{t3}) = \frac{\theta_{t3} \exp(\theta_{t1})}{\left[\theta_{t3}^2 \exp(\theta_{t1}) \times \exp(\theta_{t1})\right]^{1/2}}
$$

Multivariate Stochastic Volatility

The usual prior for the θ_{ti} series is

$$\theta_{ti} = a_i + b_i \, \theta_{t-1,i} + s_i \, z_{ti}$$

Let's call this $q(\theta_{ti} \,|\, \theta_{t-1,i})$.

Letting $\theta_t = (\theta_{t1}, \theta_{t2}, \theta_{t3})$, let,

$$q(\theta_t \,|\, \theta_{t-1}) = \Pi_{i=1}^{3} q(\theta_{ti} \,|\, \theta_{t-1,i}).$$

We usually choose the s_i so that
successive θ are not "too different".

Prior Formulation

Our prior formulation is

$$p(\theta_t \,|\theta_{t-1}) \propto q(\theta_t \mid \theta_{t-1})\; f(\theta_t).$$

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$$f(\theta_t) = \exp\left\{ \frac{-(\rho(\theta_t) - \bar{\rho})^2}{\kappa} \right\}$$

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q:
usual smoothness, don't let θ's jump around to much

f:
have preference for each θ_t, small κ means each θ_t should be such that $\rho_t \approx \bar{\rho}$

Bivariate Stochastic Volatility with Flexible Prior

$$(w_t, u_t)' \sim N(0, \Sigma(\theta_t)), \quad \theta_t = (\theta_{t1}, \theta_{t2}, \theta_{t3})$$

$$\begin{aligned} w_t &= \exp(\theta_{t1}/2)\, Z_{t1} \\ u_t &= \theta_{t3}\, w_t + \exp(\theta_{t2}/2)\, Z_{t2} \end{aligned}$$

$$\begin{aligned} p(\theta_t \,|\theta_{t-1}) &\propto q(\theta_t \,|\, \theta_{t-1})\; f(\theta_t) \\ &= q(\theta_t \,|\, \theta_{t-1})\; f(\theta_t)\; K(\theta_{t-1}) \end{aligned}$$

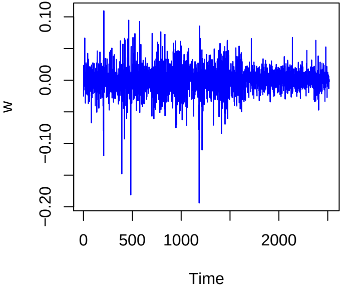
$$p(\theta_0) \propto f(\theta_0)\; \Pi_{i=1}^{3} p(\theta_{i0})$$

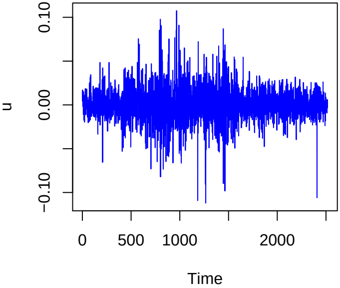
Simple Example

Today we'll just look at an example of the Bivariate model.

(not quite done coding full TVPS).

Let w and u be the observed bivariate series consisting of daily returns from two stocks in the S&P100.





Prior:

$$f(\theta_t) = \exp\left[\frac{-\left(\rho(\theta_t) - \bar{\rho}\right)^2}{\kappa}\right]$$

For this data, it is more reasonable to believe that $\rho_t > 0$!

I'll hide the details about q and show results for

$$\bar{\rho} = .8, \;\; \kappa = .01, \; .25$$

$\kappa = .01$: tight prior.

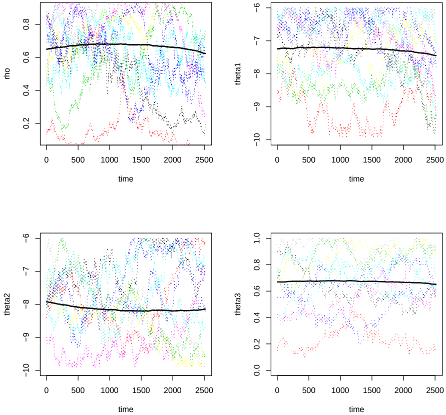
$\kappa = .25$: loose prior.

loose prior: draws from prior

black is average draw, others are individual draws

(1,1): ρ_t, (1,2): θ_{t1}

(2,1): θ_{t2}, (2,2): θ_{t3}



loose prior: draws from posterior

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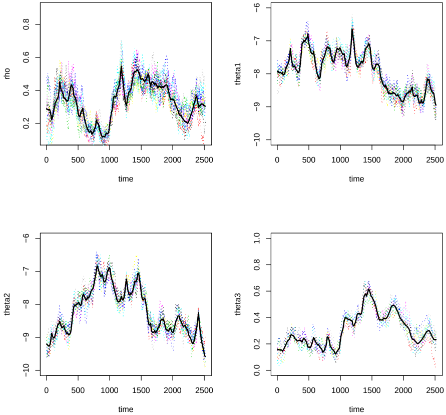

tight prior: draws from prior

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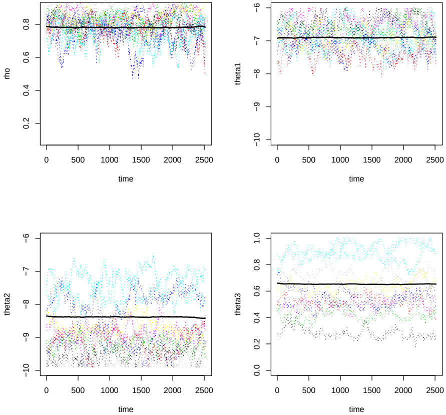

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