Put Option Implied Risk-Premia in General Equilibrium under Recursive Preferences

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Introduction

- We present a model with non-linear state-space dynamics.
- Non-standard preferences with simple dynamics present rich consumption and asset pricing implications.
- Bayesian MCMC is used to estimate structural parameters and latent states.

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Investment and Consumption

Continuous time model with one good and one representative agent.

K_t: Capital at time t.
C_t: Consumption at time t.
θ_t: expected return on production technology.

θ_t follows an OU process:

$$d\theta_t = \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta$$

Linear technology:

$$dK_t = K_t \left[\theta_t \, dt + \sigma_K \, dB_K\right] - C_t \, dt$$

where dB_θ and dB_K are correlated Brownian motion shocks with correlation $\rho > 0$.

Utility Maximization

Let J denote the value function.

Preferences: Duffie-Epstein $f(C, J)$.

Utility maximization:

$$J(K_t, \theta_t) = \sup_{C_t} E_t \left[\int_t^{\infty} f(C_s, J_s) ds \right]$$

subject to

$$\begin{aligned} dK_t &= K_t[\delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta] - C_t dt \\ d\theta_t &= \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta \end{aligned}$$

The Value Function

The value function J has the form

$$J(K_t, \theta_t) = \frac{K_t^{1-\gamma}}{1-\gamma} H(\theta_t)$$

$H(\theta_t)$ solves a non-linear second order ODE which we solve using a Chebyshev approximation.

This approximation is very accurate which implies we are very "close" to the actual ODE implied by utility maximization.

Asset Pricing

The non-linearity in the solution has important implications for asset pricing.

Let r_t^f denote the risk-free rate.

The pricing kernel follows

$$\frac{d\Lambda}{\Lambda} = -r_t^f dt - \gamma \sigma_K dB_K + \frac{H'(\theta_t)}{H(\theta_t)} \sigma_\theta dB_\theta$$

- The quantity $\frac{H'}{H}$ creates long-run risk.
- If $\ln H$ is non-linear that implies time-varying risk-premia. In this case, it creates counter-cyclical risk-premia.
- This counter-cyclicality is what we need to price insurance claims like put options.

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Endogenous Quantities

- The endogenous quantities that we want to take to the data are consumption and put option prices.
- Consumption is directly obtained, whereas option pricing is done using inverse Fourier transform.
- If $\lambda = \{\beta, \psi, \gamma, \bar{\theta}, \delta, \sigma_\theta, \rho, \sigma_K\}$ is the full set of parameters, and R_t is wealth on which put options are written

$$\begin{aligned}
\frac{dC^*}{C^*} &= \mu_C(\theta_t;\lambda)dt + \sigma_C(\theta_t;\lambda)\cdot dW \\
\frac{Put(R_t,\theta_t,\tau)}{R_t} &= \frac{X}{R_t}(Z(\theta_t) - P2(\theta_t,\tau)) - (1 - P1(\ te_t,\tau)) \\
d\theta_t &= \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta
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Inference using Put Options Data

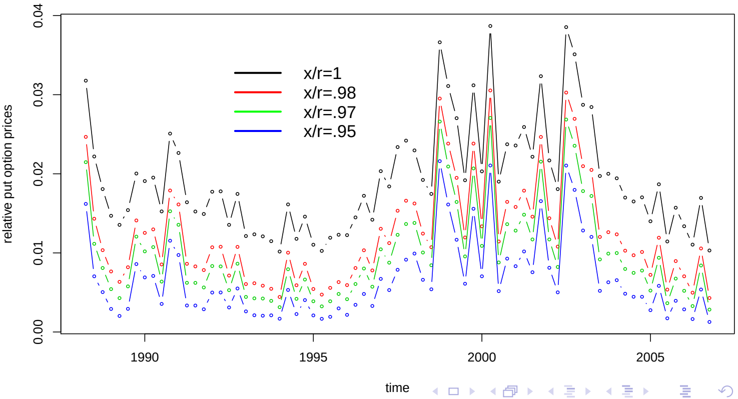
Today we just present empirical results based on put options data.

We can base inference on the joint analysis of the option data and consumption data, but the consumption data is not very informative.

Put Option Data

Data on four different options corresponding to four different strikes, measured at quarterly frequency.

- sample at third week of the first month of each quarter.
- τ (time to maturity), is one month.
- relative strikes x/r, are 1, .98,.97,.95.



Discretized Model for Option Pricing

- model parameters: $\lambda = (\beta, \psi, \gamma, \bar{\theta}, \delta, \sigma_\theta, \rho, \sigma_k)$.
- x_i is the strike, i^{th} put option.
- τ is the time till the option expires.
- P_{it}/R_t is the relative option price, i^{th} put option.

We discretize our continuous time model to obtain a nonlinear state space model for our put option prices.

Observation equation:

$$\frac{P_{it}}{R_t} = f(\theta_t, x_i/R_t, \tau, \lambda) + \sigma Z_{it}, \ i = 1, 2, 3, 4.$$

State equation:

$$\theta_t = \theta_{t-1} + \delta\,(\bar{\theta} - \theta_{t-1})\,\Delta t + \sqrt{\Delta t}\,\sigma_\theta\,Z_t.$$

Prior

We need to specify a prior for the initial state and our basic model parameters λ.

$$\theta_0 \sim N(.02, .02^2).$$

For each element of $\lambda = (\beta, \psi, \gamma, \bar{\theta}, \delta, \sigma_\theta, \rho, \sigma_k)$ our prior is flat over an interval.

But, our prior support only includes λ such that we can solve the ODE!

MCMC

For inference we use the basic Gibbs sampler:

$$\begin{aligned} \{\theta_t\} &\mid \lambda, \text{data} \\ \lambda &\mid \{\theta_t\}, \text{data} \end{aligned}$$

The draw of the states is easy
(see Hore, Johannes, Lopes, and McCulloch, and Polson (2010), "Bayesian Computation in Finance").

Basically, you just discretize the state and use FFBS (Carter & Kohn, Fruhwirth-Schnatter).

The draw of the parameters

$$\lambda \mid \{\theta_t\}, \text{data}$$

is difficult.

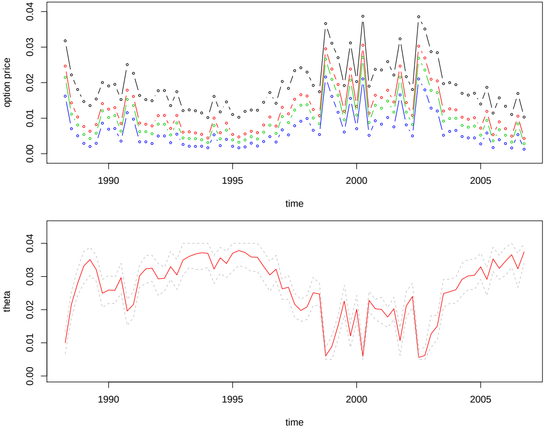
The function relating $\lambda = (\beta, \psi, \gamma, \bar{\theta}, \delta, \sigma_\theta, \rho, \sigma_k)$ to the option prices, is complex, highly non-linear, and expensive to compute.

We use a Metropolis with Gibbs, but are still working on this.

We have some fear our current inference represents a (good!) local maximum.

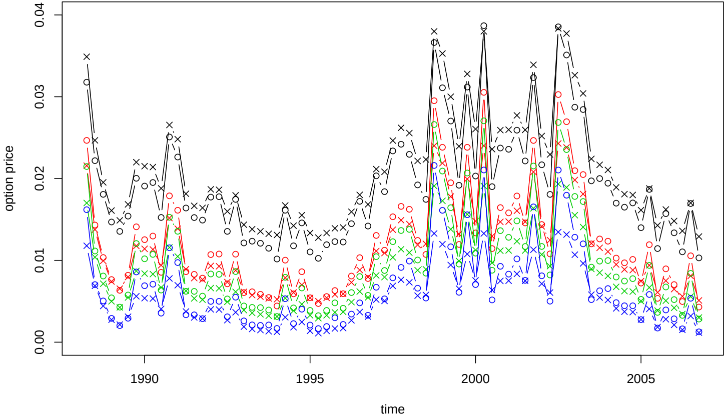
Inference for The State

- top panel: the option price data (again).
- bottom panel: red is posterior mean of θ_t, 95% (pointwise) intervals.

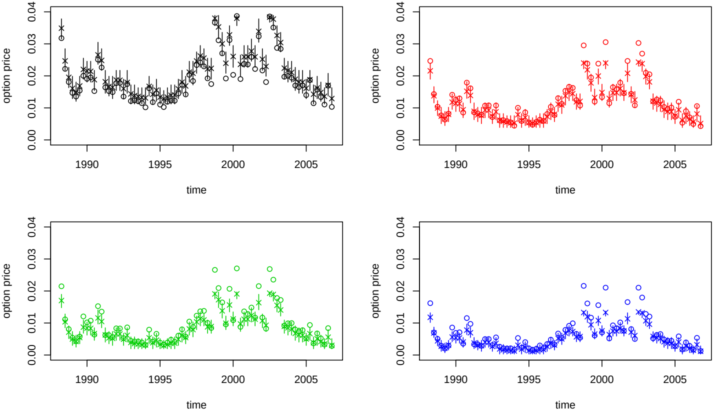


Inference for The Option Prices

- o is the data (option prices).
- x is the posterior mean.



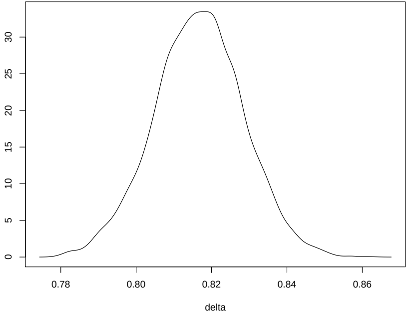
- line through x (posterior mean) is 95% posterior interval.
- a few bad misses at high prices for out-of-the-money options, but overall very good!



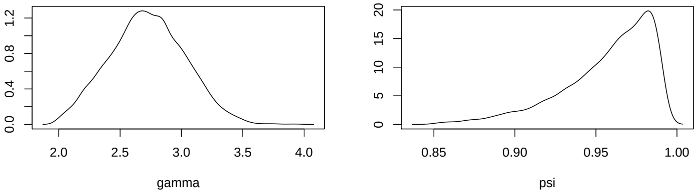
Inference for Parameters

$$\theta_t = \theta_{t-1} + \delta\,(\bar{\theta} - \theta_{t-1})\,\Delta t + \sqrt{\Delta t}\,\sigma_\theta\,Z_t.$$

Inference for δ:



Utility parameters (γ, ψ):



Conclusion

- We present a simple state-space model that has a non-linear solution.
- This non-linearity generates counter-cyclical risk-premia which is essential for option pricing.
- We present an econometric methodology to estimate latent states and parameters that take into account the existence of solution around which our theory is built.
- This methodology is very general and can be adapted to fit many non linear dynamic models in economics.

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